ARE BLACK HOLES REAL ?

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TWO NOTIONS OF REALITY

1. **MATHEMATICAL REALITY.** An object is **real** if it is mathematically consistent.
2. **PHYSICAL REALITY.** A mathematical model is **real** if it leads to effects verifiable by experiments.

Can physical reality be tested by mathematical means, in the framework of a given theory ?

EXAMPLE. Black holes are specific solutions of the Einstein field equations.

They exist as real, rich and beautiful mathematical objects, which deserve to be studied for their own sake. They are also consistent with many indirect astrophysical observations.

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WHAT IS A BLACK HOLE ?

Stationary, asymptotically flat, solutions of the Einstein field equations (in vacuum),

$$\mathrm{Ric}(g) = 0.$$

DEFINITION [External Black Hole]

- Asymptoticaly flat, globally hyperbolic, Lorentzian manifold with boundary (M, g), diffeomorphic to the complement of a cylinder $\subset \mathbb{R}^{1+3}$.
- Metric g has an asymptotically timelike, Killing vectorfield T,

$$\mathcal{L}_T g = 0.$$

- Completeness (of Null Infinity)

KERR FAMILY $\mathcal{K}(a, m)$

Boyer-Lindquist (t, r, θ, φ) coordinates.

$$-\frac{\rho^2\Delta}{\Sigma^2}(dt)^2 + \frac{\Sigma^2(\sin\theta)^2}{\rho^2}\left(d\varphi - \frac{2amr}{\Sigma^2}dt\right)^2 + \frac{\rho^2}{\Delta}(dr)^2 + \rho^2(d\theta)^2,$$

$$\begin{cases} \Delta = r^2 + a^2 - 2mr; \\ \rho^2 = r^2 + a^2(\cos\theta)^2; \\ \Sigma^2 = (r^2 + a^2)^2 - a^2(\sin\theta)^2\Delta. \end{cases}$$

Stationary. $\mathbf{T} = \partial_t$

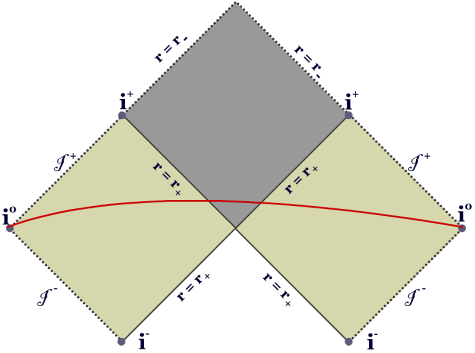
Axisymmetric. $\mathbf{Z} = \partial_\varphi$

Schwarzschild. $a = 0, m > 0$, static, spherically symmetric.

$$-\frac{\Delta}{r^2}(dt)^2 + \frac{r^2}{\Delta}(dr)^2 + r^2 d\sigma_{\mathbb{S}^2}, \qquad \frac{\Delta}{r^2} = 1 - \frac{2m}{r}$$

KERR SPACETIME $\mathcal{K}(a, m)$, $|a| \leq m$

Maximal Extension $\Delta(r_-) = \Delta(r_+) = 0, \quad \Delta = r^2 + a^2 - 2mr$

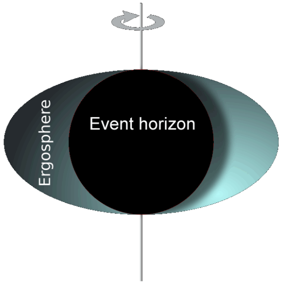


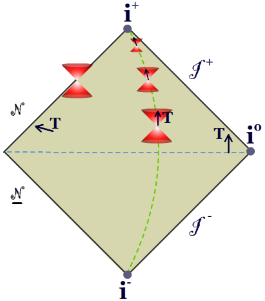
External region. $r > r_+$

Event horizon. $r = r_+$.

Black Hole. $r < r_+$

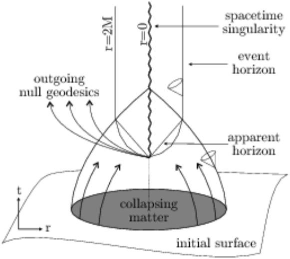
EXTERNAL KERR





- **Stationary, axisymmetric.**
- **Nonempty ergoregion**. Non- positive energy.
- **Region of trapped null geodesics**

DYNAMICAL COLLAPSE   Standard Picture



Large concentrations of matter may lead to the formation of a *dynamical* black hole settling down, by gravitational radiation, to a Kerr or Kerr-Newman stationary black hole.

PRESUPPOSES:

- Large concentrations of matter lead to the strong causal deformations of Black Holes!
- All stationary states are Kerr, or Kerr-Newman, black holes.
- These latter are stable under general perturbations.

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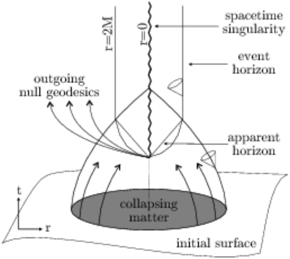

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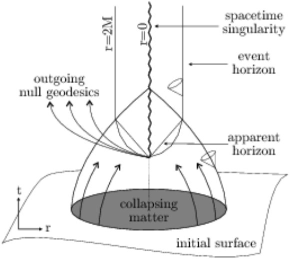

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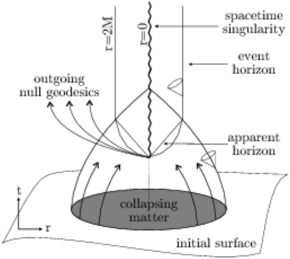

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TESTS OF REALITY $\mathrm{Ric}(g) = 0$

1. **RIGIDITY.** Does the Kerr family $\mathcal{K}(a, m)$, $0 \leq a \leq m$, exhaust all possible vacuum black holes ?
2. **STABILITY.** Is the Kerr family stable under arbitrary small perturbations ?
3. **COLLAPSE.** Can black holes form starting from reasonable initial data configurations ? Formation of trapped surfaces.

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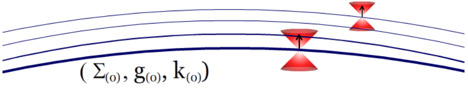
INITIAL VALUE PROBLEM: Specify initial conditions on a given initial hypersurface and study its maximal future, globally hyperbolic development. J. Leray, Y. C. Bruhat(1952)

Ric(g)=0



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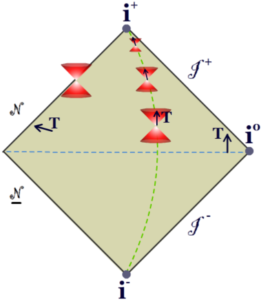

RIGIDITY CONJECTURE. Kerr family $\mathcal{K}(a, m)$, $0 \leq a \leq m$, exhaust all **stationary**, asymptotically flat, **regular** vacuum black holes.

Despite common perceptions the conjecture is far from settled!

- True in the static case. [Israel, Bunting-Masood ul Ulam]
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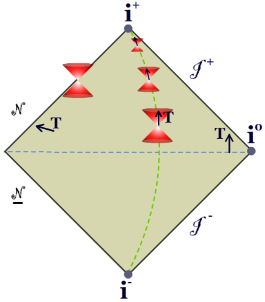

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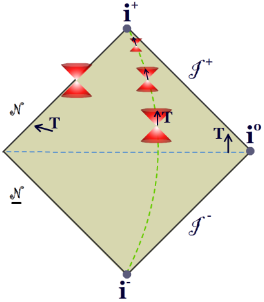

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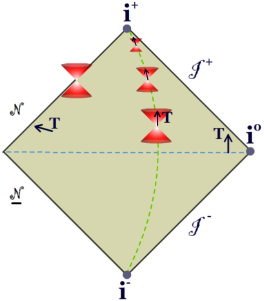

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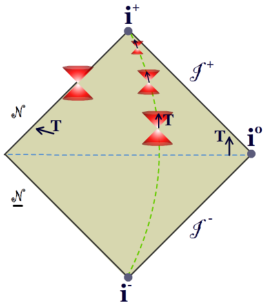

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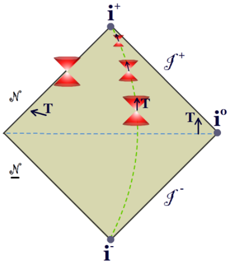
I. RIGIDITY MAIN NEW IDEAS



- There exists a second Killing v-field **H** along $\mathcal{N} \cup \underline{\mathcal{N}}$.
- Extending **H** leads to an ill posed problem.

NEW APPROACH. Design a unique continuation argument to extend **H**.

MAIN OBSTRUCTION. Presence of **T**-trapped null geodesics.

No such objects in Kerr, or close to Kerr!

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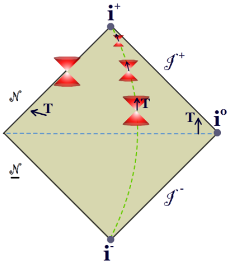

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I. RIGIDITY CONCLUSIONS

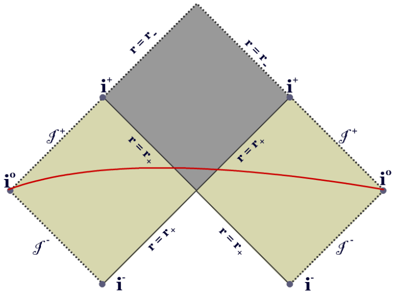
- There exist no other **explicit** stationary solutions.
- There exist no other stationary solutions close to a non extremal Kerr, (or Kerr-Newman).
- The full problem is far from being solved. Surprises are still possible for large perturbations.
- Arguments based, purely, on the continuation of the Hawking v-field **H** from the horizon are insifficient

Conjecture. [Alexakis-Ionescu-KI]. Rigidity conjecture holds true provided that there are **no T-trapped** null geodesics.

II. STABILITY

CONJECTURE[Stability of (external) Kerr].

Small perturbations of a given exterior Kerr $(\mathcal{K}(a, m),\ |a| < m)$ initial conditions have max. future developments converging to **another** Kerr solution. $\mathcal{K}(a_f, m_f)$.



STABILITY

The treatment of perturbations of Kerr spacetime has been prolixious in its complexity. Perhaps at a later time the complexity will be unravelled by deeper insights. But meantime the analysis has led into a realm of the rococo, splendorous, joyful and immensely ornate.
[S. Chandrasekhar]

According to the common perception (in Physics!) the stability problem has been solved- *based on separation of variable methods.*

- **Schwarzschild.** Regge-Wheeler(1957), Vishvevshara(1970), Zerilli(1970)
- **Kerr.** Teukolski, Press- Teukolski(1973)

Whiting(1989) Various linear equation in Kerr, including the Teukolski linearized gravity equations, have **no exponentially growing** modes.

STABILITY- FAR FROM BEING SETTLED!

If lack of exponential growing modes for the linearized equations was enough to deduce nonlinear stabilty, the presence of shock waves, extreme sensitivity to data and turbulence in fluids would be ruled out!

- Lack of exponentially growing modes is **necessary** but far from **sufficient** to establish **boundedness** of solutions to the linearized equations.
- Boundedness of solutions to the linearized equations is **necessary** but far from **sufficient** to control the nonlinear perturbations.
- One needs sufficiently strong **time decay** estimates to make sure that the nonlinear term remain negligible through the entire evolution.

STABILITY- FAR FROM BEING SETTLED!

- Precise quantitative decay estimates are often insufficient to control the nonlinear terms. The specific structure of the quadratic nonlinear terms is essential.
- Stability of the Minkowski space is trivial at the linear level and yet it has required a wealth of mathematical ideas and over 500 pages to settle.
- **weak type*** of linear **instabilities** are to be expected in view of the fact that the final Kerr solution differs from the one we perturb.

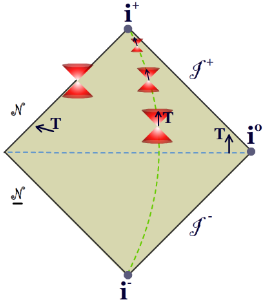
(*) Leading to lack of decay for the linearized fields

MAIN DIFFICULTIES

- Until recently even the simplest linear wave equations on fixed black holes backgrounds were not understood
- Linearized gravity system, as discussed by Teukolski and al., is not conservative. As a consequence one cannot establish, even formally, the boundedness and decay of solutions.
- The linearized gravity system must in fact have real instabilities corresponding to mass angular momentum

STABILITY OF SIMPLE LINEAR WAVES

THEOREM. The scalar wave equation $\square_g \phi = 0$ is **strongly stable** on all Kerr backgrounds $\mathcal{K}(a, m)$, $|a| < m$.



- Degeneracy of the horizon
- Ergoregion. Non-positive Energy!
- Trapped null geodesics
- Low decay at null infinity

VECTORFIELD METHOD. Flexible geometric method of deriving quantitative decay used in the stability of the Minkowski space.

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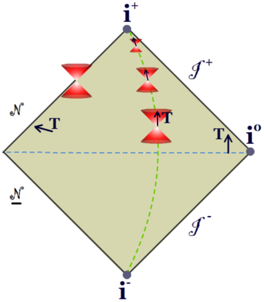

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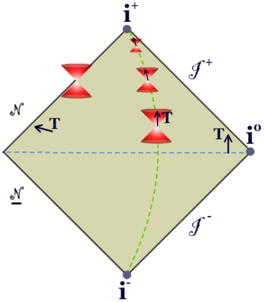

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MAIN RESULTS ON STABILITY

- Global Stability of Minkowski space [Christodoulou-Kl(1990)]
- Mode stability of the Kerr family [Whiting(1989)]
- Stability for scalar linear waves $\mathcal{K}(a, m)$, $0 \leq a << m$. [Soffer-Blue, Blue-Sterbenz, Dafermos-Rodnianski, Blue-Anderson, Tataru-Tohaneanu, etc].
- Stability for scalar linear waves $\mathcal{K}(a, m)$, $0 \leq a < m$. [Dafermos-Rodnianski-Schlapentokh Rothman (2014)]

STABILITY IN AXIAL SYMMETRY

CONJECTURE[Partial Stability]. The stability conjecture is true, at least for small, axially symmetric, perturbations of a given Kerr $\mathcal{K}(a, m)$.

WORK IN PROGRESS. Two model problems connected to the partial stability conjecture.

- Stability of Schwarzschild with respect to axially symmetric, *polarized*, perturbations.
- *Half-linear* stability of axially symmetric, perturbations of Kerr.

III. COLLAPSE

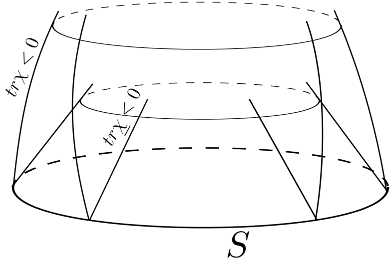
GOAL. Investigate the mechanism of formation of black holes starting with reasonable initial data configurations.

- PENROSE SINGULARITY THEOREM(1969)
- ISOTROPIC TRAPPING (CHRISTODOULOU 2008)
- NON-ISOTROPIC TRAPPING (Kl-Luk-Rodnianski(2013)

PENROSE SINGULARITY THEOREM

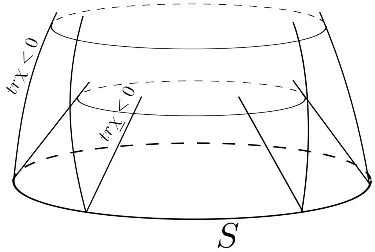
THEOREM. Space-time (M, g) cannot be future null geodesicaly complete, if

- $\text{Ric}(g)(L, L) \geq 0, \qquad \forall L$ *null*
- M contains a non-compact Cauchy hypersurface
- M contains a closed **trapped** surface S



Null expansions $\qquad \text{tr}\,\chi, \ \text{tr}\,\underline{\chi}$

MAIN IDEAS



- Show, using Raychadhouri equation, that $\partial\mathcal{J}^+(S)$ is compact

$$\frac{d}{ds}\mathrm{tr}\,\chi + \frac{1}{2}\mathrm{tr}\,\chi^2 \leq 0$$

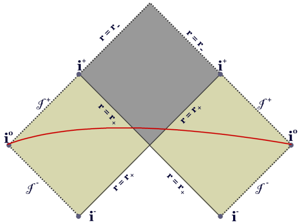
- There is a homeomorphism from $\partial\mathcal{J}^+(S)$ to its image in Σ_0.

QUESTIONS

- Can trapped surfaces form in evolution ? In vacuum ?
- Does the existence of a trapped surface implies the presence of a Black Hole ?

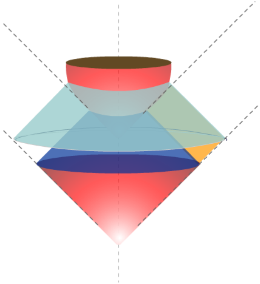
  True if **weak cosmic censorship** holds true.

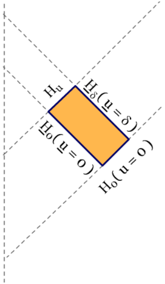
- Can singularities form starting with **non-isotropic**, initial configurations?



MAIN RESULTS

THEOREM[[Christ(2008)]. ($\exists$) open set of regular, vacuum, data whose MGFHD contains a trapped surface.



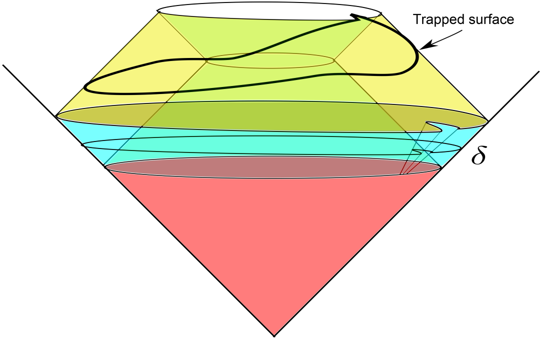


1. Specify **short pulse** characteristic data, for which one can prove a general semi-global result, with **detailed control**.
2. If, **in addition**, the data is sufficiently large, **uniformly** along all its null geodesic generators, a trapped surface must form.
3. Similar result for data given at past null infinity.

FORMATION OF TRAPPED SURFACES

THEOREM[Kl-Luk-Rodnianski(2013)] Result holds true for **non-isotropic** data concentrated near one null geodesic generator.

1. Combines all ingredients in Christodoulou's theorem with a **deformation argument** along incoming null hypersurfaces.
2. Reduces to a simple differential inequality on $S_{0,0} = H_0 \cap \underline{H}_0$.



CONCLUSIONS

1. **RIGIDITY.** Completely understood in the static case. In the stationary is only understood under additional assumptions, to insure closedness to Kerr. General case is wide open.
2. **STABILITY.** Remains wide open. We only understand the stability of Minkowski space in full. The mathematical evidence for the general stability of black holes is still scant and is essentially based on linearization. Only the so called "Poor man's linear stability" is now completel understood. There is hope that nonlinear stability in the restrictive class of axial symmetric perturbations could be settled in the near future.
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